Interpreting LCTRSs in TRSs

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Logically Constrained Term Rewriting Systems (LCTRSs)

[Kop & Nishida,FroCoS 2013]

$$\mathcal{R} = \left\{ \begin{array}{lcl} \mathsf{sum}(x) & \to & 0 \ [0 \geq x] \\ \mathsf{sum}(x) & \to & x + \mathsf{sum}(x + -1) \ [\neg(0 \geq x)] \end{array} \right\}$$

- (many-sorted) theory signature $\Sigma_{\mathsf{th}} = \langle \mathcal{S}_{\mathsf{th}}, \mathcal{F}_{\mathsf{th}} \rangle$ and term signature $\Sigma_{\mathsf{te}} = \langle \mathcal{S}_{\mathsf{te}}, \mathcal{F}_{\mathsf{te}} \rangle$
- for $f : \tau_1 \times \cdots \times \tau_n \to \tau_0 \in \mathcal{F}_{\mathsf{th}}$, we ask $\tau_0, \ldots, \tau_n \in \mathcal{S}_{\mathsf{th}}$.
- An underlying model (background theory) $\mathcal{M}$ over Σ_{th} is given, e.g. $\mathbb{B}, \mathbb{Z}, \wedge, +, \ldots$
- All elements of carrier set $|\mathcal{M}|$ are supposed to exist in Σ_{th} as contants (which we call *values*), e.g. $\mathsf{true}, \mathsf{false}, 0, -256, \ldots$
- A rule has form $\ell \to r \ [\varphi]$, where φ is a Σ_{th}-term of type Bool and $\mathrm{root}(\ell) \in \mathcal{F}_{\mathsf{te}}$.
- Calculations by operations in $\mathcal{M}$ is embodied: e.g.

  $1 + 1 \to 2, \quad 12 \geq 10 \to \mathsf{true}, \quad \mathsf{true} \wedge \mathsf{false} \to \mathsf{false}, \ldots$

Rewrite Steps of LCTRSs (1)

$$\mathcal{R} = \left\{ \begin{array}{lcll} \mathsf{sum}(x) & \to & 0 & [0 \geq x] \\ \mathsf{sum}(x) & \to & x + \mathsf{sum}(x + -1) & [\neg(0 \geq x)] \end{array} \right\}$$

(over the integer arithmetic)

- Rule Step ($\to_{\mathsf{rule}}$): rewriting using given rewrite rules
- The rule $\ell \to r\ [\varphi]$ is applied when the constraint φ is satisfied. (Evaluation of constraint is a meta-calculation.)
- Calculation Step ($\to_{\mathsf{calc}}$): rewriting induced by the underlying model
- Each calculation step is applied for the term $f(v_1, \ldots, v_n)$ with $f \in \mathcal{F}_{\mathsf{th}}$ and values $v_1, \ldots, v_n$.

$$\begin{array}{rcl} \underline{\mathsf{sum}(1)} & \to_{\mathsf{rule}} & 1 + \mathsf{sum}(\underline{1 + -1}) \\ & \to_{\mathsf{calc}} & 1 + \underline{\mathsf{sum}(0)} \\ & \to_{\mathsf{rule}} & \underline{1 + 0} \\ & \to_{\mathsf{calc}} & 1 \end{array}$$

Rewrite Steps of LCTRSs (2)

$$\mathcal{R} = \left\{ \begin{array}{llll} \mathsf{minus}(x, y) & \to & z & [x = y + z] \\ \mathsf{inc}(x) & \to & x + 1 & \\ \Omega(x) & \to & \Omega(y) & \end{array} \right\}$$

- Do we have: $\mathsf{minus}(5, 2) \to_{\mathsf{rule}} 3$? ... YES
- Do we have: $\mathsf{minus}(5, 2) \to_{\mathsf{rule}} 5 - 2$? ... NO
- Do we have: $\mathsf{minus}(x, y) \to_{\mathsf{rule}} x - y$? ... NO
- Do we have: $\mathsf{minus}(x + 1, 1) \to_{\mathsf{rule}} x$? ... NO
- Do we have: $\mathsf{inc}(x - 1) \to_{\mathsf{rule}} (x - 1) + 1$? ... YES
- Do we have: $\Omega(1) \to_{\mathsf{rule}} \Omega(2)$? ... YES
- Do we have: $\Omega(x + 1) \to_{\mathsf{rule}} \Omega(x + 2)$? ... NO
- Do we have: $\Omega(x + 1) \to_{\mathsf{rule}} \Omega(2)$? ... YES

Instantiation of logical variables are restricted to values.

$$\mathcal{LV}ar(\ell \to r\ [\varphi]) = \mathcal{V}(\varphi) \cup (\mathcal{V}(r) \setminus \mathcal{V}(\ell))$$

Definition of Rewrite Steps

Suppose that signature $\Sigma_{\mathsf{th}} = \langle \mathcal{S}_{\mathsf{th}}, \mathcal{F}_{\mathsf{th}} \rangle$, $\Sigma_{\mathsf{te}} = \langle \mathcal{S}_{\mathsf{te}}, \mathcal{F}_{\mathsf{te}} \rangle$, Σ_{th}-structure $\mathcal{M}$, and rewrite rules $\mathcal{R}$ are given.

1. (rule step)

$$s \to_{\mathsf{rule}} t$$

   if $s = C[\ell\sigma]$ and $t = C[r\sigma]$ for some context C, rewrite rule $\rho : \ell \to r\ [\varphi] \in \mathcal{R}$, and substitution σ such that
   - $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}al$, and
   - $\models_{\mathcal{M}} \varphi\sigma$ (or equivalently, $\models_{\mathcal{M},\sigma} \varphi$)

2. (calculation step)

$$s \to_{\mathsf{calc}} t$$

   if $s = C[f(v_1, \ldots, v_n)]$ and $t = C[v_0]$ for some context C, $f \in \mathcal{F}_{\mathsf{th}}$, $v_0, v_1, \ldots, v_n \in \mathcal{V}al$ such that $f^{\mathcal{M}}(v_1, \ldots, v_n) = v_0$.

Interpreting LCTRSs by TRSs (1)

[Mitterwallner et al., IWC 2023]

- Simulation of calculation steps
  $\Rightarrow$ provide all underlying operations of $\mathcal{M}$ as rewrite rules.

$$\mathsf{rs}(\mathcal{M}) = \{\ f(v_1, \ldots, v_n) \to v_0 \mid f \in \mathcal{F}_{\mathsf{th}}, v_0, \ldots, v_n \in \mathcal{V}al, f^{\mathcal{M}}(v_1, \ldots, v_n) = v_0\ \}$$

Proposition

$s \to_{\mathsf{calc}} t$ (in LCTRSs) iff $s \to_{\mathsf{rs}(\mathcal{M})} t$ (in TRSs).

Proof. ($\Rightarrow$) Let $s = C[f(v_1, \ldots, v_n)]$, $t = C[v_0]$ with $f \in \mathcal{F}_{\mathsf{th}}$, $v_0, \ldots, v_n \in \mathcal{V}al$ such that $f^{\mathcal{M}}(v_1, \ldots, v_n) = v_0$. Then $f(v_1, \ldots, v_n) \to v_0 \in \mathsf{rs}(\mathcal{M})$. Thus, $s = C[f(v_1, \ldots, v_n)] \to_{\mathsf{rs}(\mathcal{M})} C[v_0] = t$.
($\Leftarrow$) Let $s = C[\ell\sigma]$, $t = C[r\sigma]$ with $\ell \to r \in \mathsf{rules}(\mathcal{M})$. Then, by definition $\ell = f(v_1, \ldots, v_n)$ and $r = v_0$ for some $f \in \mathcal{F}_{\mathsf{th}}$, $v_0, \ldots, v_n \in \mathcal{V}al$ such that $f^{\mathcal{M}}(v_1, \ldots, v_n) = v_0$. Thus, $s = C[\ell] = C[f(v_1, \ldots, v_n)]$ and $t = C[r] = C[v_0]$. By $f^{\mathcal{M}}(v_1, \ldots, v_n) = v_0$, we have $s \to_{\mathsf{calc}} t$. $\square$

Interpreting LCTRSs by TRSs (2)

[Mitterwallner et al., IWC 2023]

- Simulation of rule steps
  $\Rightarrow$ provide all instantiation of rules by $\sigma : \mathcal{LV}ar(\rho) \to \mathcal{V}al$ satisfying $\models_{\mathcal{M}} \varphi\sigma$.

$$\overline{\mathcal{R}} = \bigcup_{\rho:\ \ell \to r[\varphi] \in \mathcal{R}} \{\ l\sigma \to r\sigma \mid \sigma : \mathcal{LV}ar(\rho) \to \mathcal{V}al, \models_{\mathcal{M}} \varphi\sigma\ \}$$

Proposition

$s \to_{\mathsf{rule}} t$ (in LCTRSs) iff $s \to_{\overline{\mathcal{R}}} t$ (in TRSs).

Proof. ($\Rightarrow$) Let $s = C[\ell\sigma]$, $t = C[r\sigma]$ with $\rho :\ \ell \to r[\varphi] \in \mathcal{R}$. Take $\sigma_{\mathsf{v}} = \sigma\lfloor(\mathcal{LV}ar(\rho))$, $\sigma' = \sigma\lfloor(\mathcal{LV}ar(\rho))^{\mathrm{c}}$. By $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}al$, we have $\sigma_{\mathsf{v}} : \mathcal{LV}ar(\rho) \to \mathcal{V}al$, $\sigma = \sigma' \circ \sigma_{\mathsf{v}}$, $\models_{\mathcal{M}} \varphi\sigma_{\mathsf{v}}$; so, $l\sigma_{\mathsf{v}} \to r\sigma_{\mathsf{v}} \in \overline{\mathcal{R}}$. Thus, $s = C[\ell\sigma] = C[(\ell\sigma_{\mathsf{v}})\sigma'] \to_{\overline{\mathcal{R}}} C[(r\sigma_{\mathsf{v}})\sigma'] = C[r\sigma] = t$. ($\Leftarrow$) Let $s = C[(\ell\sigma)\theta]$ $t = C[(r\sigma)\theta]$ with $\ell\sigma \to r\sigma \in \overline{\mathcal{R}}$ and $\rho : \ell \to r\ [\varphi] \in \mathcal{R}$. As $\sigma : \mathcal{LV}ar(\rho) \to \mathcal{V}al$, $\mathcal{V}(\ell\sigma, r\sigma) \subseteq (\mathcal{LV}ar(\rho))^{\mathrm{c}}$, take $\theta' = \theta\lfloor(\mathcal{LV}ar(\rho))^{\mathrm{c}}$, and we have $\ell(\sigma \uplus \theta') = (\ell\sigma)\theta' = (\ell\sigma)\theta$ and $r(\sigma \uplus \theta') = (r\sigma)\theta' = (r\sigma)\theta$. By $\mathcal{V}(\varphi) \subseteq \mathcal{LV}ar(\rho)$, $\models_{\mathcal{M}} \varphi(\sigma \uplus \theta')$. So, $s = C[\ell(\sigma \uplus \theta')] \to_{\mathsf{rule}} C[r(\sigma \uplus \theta')] = t$.

Example.

$$\mathcal{R} = \left\{ \begin{array}{lll} \mathsf{minus}(x, y) & \rightarrow & z \qquad [x = y + z] \\ \mathsf{inc}(x) & \rightarrow & x + 1 \\ \Omega(x) & \rightarrow & \Omega(y) \end{array} \right\}$$

$$\overline{\mathcal{R}} = \left\{ \begin{array}{llllll} \mathsf{minus}(0, 0) & \rightarrow & 0, & \mathsf{minus}(1, 0) & \rightarrow & 1, \\ \mathsf{minus}(0, 1) & \rightarrow & -1, & \dots\dots & & \\ \mathsf{inc}(x) & \rightarrow & x + 1, & & & \\ \Omega(x) & \rightarrow & \Omega(0), & \Omega(x) & \rightarrow & \Omega(1), \\ \Omega(x) & \rightarrow & \Omega(-1), & \dots\dots & & \end{array} \right\}$$

$$\begin{array}{lll} \mathsf{minus}(5, 2) & \rightarrow_{\mathsf{rule}} & 3 \\ \mathsf{inc}(x - 1) & \rightarrow_{\mathsf{rule}} & (x - 1) + 1 \\ \Omega(1) & \rightarrow_{\mathsf{rule}} & \Omega(2) \\ \Omega(x + 1) & \rightarrow_{\mathsf{rule}} & \Omega(2) \end{array} \qquad \begin{array}{lll} \mathsf{minus}(5, 2) & \not\rightarrow_{\mathsf{rule}} & 5 - 2 \\ \mathsf{minus}(x, y) & \not\rightarrow_{\mathsf{rule}} & x - y \\ \mathsf{minus}(x + 1, 1) & \not\rightarrow_{\mathsf{rule}} & x \\ \Omega(x + 1) & \not\rightarrow_{\mathsf{rule}} & \Omega(x + 2) \end{array}$$

Rewriting on Contrained Terms

[Kop & Nishida,FroCoS 2013]

Three ingredients: $s[\pi] \sim t[\psi]$, $s[\pi] \rightarrow_{\mathsf{calc}} t[\psi]$, and $s[\pi] \rightarrow_{\mathsf{rule}} t[\psi]$.

1. $$s[\pi] \sim t[\psi]$$

   if

   - $\forall\gamma$: respecting $s[\pi]$, $\exists\delta$: respecting $t[\psi]$ such that $s\gamma = t\delta$.
   - $\forall\delta$: respecting $t[\psi]$, $\exists\gamma$: respecting $s[\pi]$ such that $t\delta = s\gamma$.

   γ respects $s[\pi] \Leftrightarrow \{\gamma(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al$ and $\models_{\mathcal{M}} \pi\gamma$
   δ respects $t[\psi] \Leftrightarrow \{\delta(x) \mid x \in \mathcal{V}(\psi)\} \subseteq \mathcal{V}al$ and $\models_{\mathcal{M}} \psi\delta$

2. $$s[\pi] \rightarrow_{\mathsf{calc}} t[\psi]$$

   if

   - $s = C[f(s_1, \dots, s_n)]$ with $f \in \mathcal{F}_{\mathsf{th}}$, $s_1, \dots, s_n \in \mathcal{V}(\pi) \cup \mathcal{V}al$,
   - $t = C[x]$ with x: fresh variable
   - $\psi = (\pi \wedge x = f(s_1, \dots, s_n))$

3.

$$s[\pi] \to_{\mathsf{rule}} t[\psi]$$

if

- π is satisfiable and $\psi = \pi$.
- $s = C[\ell\sigma]$and $t = C[r\sigma]$ with $\rho : \ell \to r\ [\varphi] \in \mathcal{R}$
- $\mathrm{Dom}(\sigma) = \mathcal{V}(\ell, r, \varphi)$
- $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}(\pi) \cup \mathcal{V}al$
- $\models_{\mathcal{M}} (\pi \Rightarrow \varphi\sigma)$

How can we interpret rewriting on contrained terms?

Interpreting Contrained Terms

Natural(?) idea:

$$\begin{array}{rcl} [\![\ s[\pi] \ ]\!] & = & \{s\gamma \mid \{\gamma(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al, \models_{\mathcal{M}} \pi\gamma\} \\ & (= & \{s\gamma \mid \gamma \text{ respects } s[\pi]\}) \end{array}$$

Example.

$$\begin{array}{rcl} [\![\ x + y \ [x \geq 0] \ ]\!] & = & \{0 + y, 1 + (y + 1), 1 + (x + y), \ldots\} \\ & = & \{n + t \mid n \in \mathbb{Z}, t \in \mathrm{T}(\mathcal{F}, \mathcal{V})^{\mathsf{Int}}\} \end{array}$$

Theorem

$s[\pi] \sim t[\psi]$ iff $[\![\ s[\pi] \ ]\!] = [\![\ t[\psi] \ ]\!]$.

Proof. It suffices to show that the following two are equivalent:

1. $\forall\gamma$: respecting $s[\pi]$, $\exists\delta$: respecting $t[\psi]$ such that $s\gamma = t\delta$
2. $[\![\ s[\pi] \ ]\!] \subseteq [\![\ t[\psi] \ ]\!]$

$(1 \Rightarrow 2)$ Suppose $u \in [\![\ s[\pi] \ ]\!]$. Then $u = s\gamma$ for some γ that respects $s[\pi]$. Then, there exists δ respecting $t[\psi]$ such that $s\gamma = t\delta$. Thus, there exists δ respecting $t[\psi]$ such that $u = t\delta$. Hence, $u \in [\![\ t[\psi] \ ]\!]$.
$(2 \Rightarrow 1)$ Suppose that γ respects $s[\pi]$. Then $s\gamma \in [\![\ s[\pi] \ ]\!]$. Thus, $s\gamma \in [\![\ t[\psi] \ ]\!]$. Then, there exists δ respecting $t[\psi]$ such that $s\delta = t\delta$. □

Interpreting Calculation Steps on Constrained Terms

Lemma(?)

$s[\pi] \to_{\mathsf{calc},p} t[\psi]$ iff $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathsf{calc},p} u'\} = [\![\ t[\psi]\ ]\!]$.

Proof. ($\Rightarrow$) Suppose $s = C[f(s_1, \dots, s_n)]_p$ with $f \in \mathcal{F}_{\mathsf{th}}$ $s_1, \dots, s_n \in \mathcal{V}(\pi) \cup \mathcal{V}al$, and $t = C[x]_p$ with x: fresh variable, and $\psi = (\pi \wedge x = f(s_1, \dots, s_n))$.
We now show $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathsf{calc},p} u'\} = [\![\ t[\psi]\ ]\!]$.
($\subseteq$) Let $u \in [\![\ s[\pi]\ ]\!]\}$. Then, $u = s\gamma$ for some γ respecting π. We have $u|_p = (s\gamma)|_p = (s|_p)\gamma = f(s_1, \dots, s_n)\gamma = f(s_1\gamma, \dots, s_n\gamma)$.
Since $s_1, \dots, s_n \in \mathcal{V}(\pi) \cup \mathcal{V}al$, and $\{\gamma(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al$, $s_1\gamma, \dots, s_n\gamma \in \mathcal{V}al$. Thus, $u \to_{\mathsf{calc},p} u[v]_p = u'$ with $v = f^{\mathcal{M}}(s_1\gamma, \dots, s_n\gamma)$.
Take δ such that $\delta(x) = v$ and $\delta(y) = \gamma(y)$ for $y \neq x$. Then $t\delta = C[x]_p\delta = C\gamma[v]_p = s\gamma[v]p = u[v]_p$. Also, by $x \notin \mathcal{V}(\psi)$, we have $\pi\gamma = \pi\delta$. Furthermore,
$\delta(x) = v = f^{\mathcal{M}}([\![s_1\delta]\!]_{\mathcal{M}}, \dots, [\![s_n\delta]\!]_{\mathcal{M}}) = [\![f(s_1, \dots, s_n)\gamma]\!]_{\mathcal{M}}$.

Thus, $\models_{\mathcal{M}} (\pi \wedge x = f(s_1, \ldots, s_n))\delta$. Hence, δ respects $t[\psi]$ and $u' = t\delta$. Hence, $u' \in [\![\ t[\psi] \ ]\!]$.
($\supseteq$) Suppose $w \in [\![\ t[\psi] \ ]\!]$. Then $w = t\delta$ for some δ respecting $t[\delta]$. Thus, $\{\delta(x) \mid x \in \mathcal{V}(\psi)\} \subseteq \mathcal{V}al$ and $\models_{\mathcal{M}} \psi\delta$. As $t = C[x]_p$ with x: fresh variable, $t\delta = C\delta[\delta(x)]_p$. We now show $u \to_{\mathsf{rule},p} w$ for some $u \in [\![\ s[\pi] \ ]\!]$.
Firstly, as $\psi = (\pi \wedge x = f(s_1, \ldots, s_n))$, we have $\models_{\mathcal{M}} \pi\delta$ and $\mathcal{V}(\pi) \subseteq \mathcal{V}(\psi)$. Thus, by $\{\delta(x) \mid x \in \mathcal{V}(\psi)\} \subseteq \mathcal{V}al$, we have $\{\delta(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al$. Together with $\models_{\mathcal{M}} \pi\delta$, we obtain that δ respects π.
Moreover, we have $\models_{\mathcal{M}} \delta(x) = f(s_1\delta, \ldots, s_n\delta)$, i.e. $\delta(x) = [\![\delta(x)]\!]_{\mathcal{M}} = f^{\mathcal{M}}([\![s_1\delta]\!]_{\mathcal{M}}, \ldots, [\![s_n\delta]\!]_{\mathcal{M}}) = [\![f(s_1, \ldots, s_n)\delta]\!]_{\mathcal{M}}$.
Now, take $u = w[f(s_1, \ldots, s_n)\delta]_p$. Since $s_1\delta, \ldots, s_n\delta \in \mathcal{V}al$, and $f^{\mathcal{M}}(s_1\delta, \ldots, s_n\delta) = [\![f(s_1, \ldots, s_n)\delta]\!]_{\mathcal{M}} = [\![u|_p]\!]_{\mathcal{M}}$, we have $u \to_{\mathsf{rule},p} u[\delta(x)] = w[\delta(x)] = w$.
Then, $u = w[f(s_1, \ldots, s_n)\delta]_p = t\delta[f(s_1, \ldots, s_n)\delta]_p = t[f(s_1, \ldots, s_n)]_p\delta = C[f(s_1, \ldots, s_n)]_p\delta = s\delta$. Hence, $u = s\delta$ and δ respects π. Thus, $u \in [\![\ s[\pi] \ ]\!]$.

($\Leftarrow$?)

Counterexample (1).

Let $s[\pi] = +(x, x)[x = 0 \vee x = 1]$ and $t[\psi] = y[y = 0 \vee y = 2]$.
Then, $[\![\ s[\pi]\ ]\!] = \{+(0, 0), +(1, 1)\}$.
Thus, $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathsf{calc},\epsilon} u'\} = \{0, 2\} = [\![\ t[\psi]\ ]\!]$.
But $s\ [\pi] \not\to_{\mathsf{calc},\epsilon} t\ [\psi]$.
Here,we only have

$$\begin{array}{lll} s[\pi] & \to_{\mathsf{calc}} & y[(x = 0 \vee x = 1) \wedge y = +(x, x)] \\ & \sim & y[y = 0 \vee y = 2] \end{array}$$

Counterexample (2).

Let $s[\pi] = +(x, x)[x \neq x]$ and $t[\psi] = +(x, y)[x \neq x \wedge y \neq y]$.
Then, $[\![\ s[\pi]\ ]\!] = [\![\ t[\psi]\ ]\!] = \emptyset$, and thus,
$\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathsf{calc},\epsilon} u'\} = \emptyset = [\![\ t[\psi]\ ]\!]$.
But $s\ [\pi] \not\to_{\mathsf{calc},\epsilon} t\ [\psi]$.

Lemma

Suppose

- π is satisfiable, $p \in \mathrm{Pos}(s)$,
- for any $u \in [\![\ s[\pi]\ ]\!]$ there exists u' such that $u \to_{\mathsf{calc},p} u'$, and
- $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathsf{calc},p} u'\} = [\![\ t[\psi]\ ]\!]$.

Then, $s[\pi] \to_{\mathsf{calc},p} \circ \sim t[\psi]$.

Proof. By satisfiability, $[\![\ s[\pi]\ ]\!] \neq \emptyset$. Thus, there exists $u \in [\![\ s[\pi]\ ]\!]$ and u', such that $u \to_{\mathsf{calc},p} u'$. Thus, $u = C[f(u_1, \ldots, u_n)]_p$ for some $f \in \mathcal{F}_{\mathsf{te}}$, and $u_1, \ldots, u_n \in \mathcal{V}al$.
By $u \in [\![\ s[\pi]\ ]\!]$, $u = s\gamma$ for some γ such that γ respects π. Thus, $s = \hat{C}[f(s_1, \ldots, s_n)]_p$ with $\hat{C}\gamma = C$ and $s_i\gamma = u_i$ $(1 \leq i \leq n)$.
Suppose $s_i \notin \mathcal{V}al$. If $s_i \notin \mathcal{V}(\pi)$, then one can modify γ such as $s_i\gamma \notin \mathcal{V}al$, while keep respecting π. This contradicts our second condition. Thus, $s_i \in \mathcal{V}(\pi) \cup \mathcal{V}al$ for $i = 1, \ldots, n$.
Thus, $s\ [\pi] \to_{\mathsf{calc},p} s[x]_p\ [\pi \wedge x = f(s_1, \ldots, s_n)]$. It remains to show $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathsf{calc},p} u'\} = [\![\ s[x]_p\ [\pi \wedge x = f(s_1, \ldots, s_n)]\ ]\!]$. But this follows as $s|_p = f(s_1, \ldots, s_n)$.

Interpreting Calculation Steps on Constrained Terms

So, we have

Theorem

If $s[\pi] \to_{\mathsf{calc},p} t[\psi]$, then
$\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathrm{rs}(\mathcal{M}),p} u'\} = [\![\ t[\psi]\ ]\!]$.

Theorem

Suppose

- π is satisfiable, $p \in \mathrm{Pos}(s)$,
- for any $u \in [\![\ s[\pi]\ ]\!]$ there exists u' such that $u \to_{\mathrm{rs}(\mathcal{M}),p} u'$, and
- $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\mathrm{rs}(\mathcal{M}),p} u'\} = [\![\ t[\psi]\ ]\!]$.

Then, $s[\pi] \to_{\mathsf{calc},p} \circ \sim t[\psi]$.

What is the precise correspondence? Bisimilarity? Functor?

Interpreting Rule Steps on Constrained Terms ...

At this point, I remind that [Kop & Nishida,FroCoS 2013] already shows

Proposition [Kop & Nishida, FroCoS 2013]

If $s[\pi] \to t[\psi]$ then for any γ that respect π there exists δ that respect ψ such that $s\gamma \to t\psi$.

In our terminology, this is equivalent to:

Proposition

If $s[\pi] \to t[\psi]$ then $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to u'\} \subseteq [\![\ t[\psi]\ ]\!]$.

The following our version is slightly stronger than this (?).

Conjecture

If $s[\pi] \to t[\psi]$ then $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to u'\} = [\![\ t[\psi]\ ]\!]$.

Interpreting Rule Steps on Constrained Terms

Lemma

If $s[\pi] \to_{\mathsf{rule},p} t[\pi]$, then $\{u' \mid u \in [\![\, s[\pi] \,]\!], u \to_{\mathsf{rule},p} u'\} = [\![\, t[\pi] \,]\!]$.

Proof. Suppose π is satisfiable, $s = C[\ell\sigma]_p$ and $t = C[r\sigma]_p$, with $\rho : \ell \to r\ [\varphi] \in \mathcal{R}$, and $\mathrm{Dom}(\sigma) = \mathcal{V}(\ell, r, \varphi)$, and $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}(\pi) \cup \mathcal{V}al$, and $\models_{\mathcal{M}} (\pi \Rightarrow \varphi\sigma)$.
We now show $\{u' \mid u \in [\![\, s[\pi] \,]\!], u \to_{\mathsf{rule},p} u'\} = [\![\, t[\pi] \,]\!]$.
($\subseteq$) Suppose $u \in [\![\, s[\pi] \,]\!]$. Then, $u = s\gamma$ with γ respecting π. Thus, $\models_{\mathcal{M}} \pi\gamma$ and $\{\gamma(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al$. Also, by $s[\pi] \to_{\mathsf{rule},p} t[\pi]$, we have $u|_p = s|_p\gamma = (\ell\sigma)\gamma$. Since $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}(\pi) \cup \mathcal{V}al$ and $\{\gamma(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al$, we have $\{\gamma(\sigma(x)) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}al$. By $\models_{\mathcal{M}} (\pi \Rightarrow \varphi\sigma)$, we have $\models_{\mathcal{M}} (\pi\gamma \Rightarrow \varphi\sigma\gamma)$, and hence by $\models_{\mathcal{M}} \pi\gamma$, we have $\models_{\mathcal{M}} \varphi\sigma\gamma$. Thus,
$u = s\gamma = C[\ell\sigma]\gamma = C\gamma[\ell\sigma\gamma] \to_{\mathsf{rule}} C\gamma[r\sigma\gamma]$. Let $u' = C\gamma[r\sigma\gamma]$. Since $t = C[r\sigma]_p$, we have $u' = C\gamma[r\sigma\gamma] = C[r\sigma]\gamma = t\gamma$. Since γ respects π, it follows $u' \in [\![\, t[\pi] \,]\!]$.

($\supseteq$)
Suppose $w \in [\![\ t[\pi]\ ]\!]$. Then, $w = t\delta$ with δ respecting π. Thus, $\models_{\mathcal{M}} \pi\delta$ and $\{\delta(x) \mid x \in \mathcal{V}(\pi)\} \subseteq \mathcal{V}al$. Also, by $s[\pi] \to_{\mathsf{rule},p} t[\pi]$, we have $w|_p = t|_p\delta = (r\sigma)\delta$.
Since $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}(\pi) \cup \mathcal{V}al$ and $\mathcal{V}(\pi) \subseteq \mathcal{LV}ar(\rho)$, we have $\{\delta(\sigma(x)) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}al$. By $\models_{\mathcal{M}} (\pi \Rightarrow \varphi\sigma)$, we have $\models_{\mathcal{M}} (\pi\delta \Rightarrow \varphi\sigma\delta)$, and hence by $\models_{\mathcal{M}} \pi\delta$, we have $\models_{\mathcal{M}} \varphi\sigma\delta$.
Also, $w = t\delta = C[r\sigma]\delta = C\delta[r\sigma\delta]$. Take $u = C\delta[\ell\sigma\delta]$. Then, $u = C\delta[\ell\sigma\delta] \to_{\mathsf{rule},p} C\delta[r\sigma\delta] = w$.
Since $s = C[\ell\sigma]_p$, we have $u = C\delta[\ell\sigma\delta] = C[\ell\sigma]\gamma = s\gamma$. Since γ respects π, it follows $u \in [\![\ s[\pi]\ ]\!]$. $\square$

Conjecture

Suppose

- π is satisfiable, $p \in \mathrm{Pos}(s)$, $\rho \in \mathcal{R}$,
- for any $u \in [\![s[\pi]]\!]$ there exists u' such that $u \to_{\rho,p} u'$, and
- $\{u' \mid u \in [\![s[\pi]]\!], u \to_{\rho,p} u'\} = [\![t[\pi]]\!]$.

Then, $s[\pi] \to_{\mathsf{rule},p} t[\pi]$.

Proof. Let $\rho : \ell \to r \; [\varphi] \in \mathcal{R}$. By satisfiability, $[\![s[\pi]]\!] \neq \emptyset$. Thus, there exists $u \in [\![s[\pi]]\!]$ and u', such that $u \to_{\rho,p} u'$. Thus, $u = C[\ell\sigma]_p$, $u' = C[r\sigma]_p$, $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}al$, and $\models_{\mathcal{M}} \varphi\sigma$.
By $u \in [\![s[\pi]]\!]$, $u = s\gamma$ for some γ such that γ respects π.

Thus, by $u = s\gamma$ and $u = C[\ell\sigma]_p$, we know $s = \hat{C}[\hat{\ell}\hat{\sigma}]_p$, $\hat{C}\gamma = C$ and $(\hat{\ell}\hat{\sigma})\gamma = \ell\sigma$??? ...If $\hat{\ell} \neq \ell$ then we can not rewrite $s[\pi]$...

Counterexample.

$$\mathcal{R} = \{\rho : \mathsf{f}(0) \to 1\}$$

Take $s[\pi] = \mathsf{f}(x)[x = 0]$ and $t[\pi] = 1[x = 0]$. Then, $[\![s[\pi]]\!] = \{\mathsf{f}(0)\}$ and $[\![t[\pi]]\!] = \{1\}$. Take $p = \epsilon$. Then,

- π is satisfiable✓, $p \in \mathrm{Pos}(s)$✓, $\rho \in \mathcal{R}$✓,
- for any $u \in [\![s[\pi]]\!]$ there exists u' such that $u \to_{\rho,p} u'$✓, and
- $\{u' \mid u \in [\![s[\pi]]\!], u \to_{\rho,p} u'\} = [\![t[\pi]]\!]$✓.

But we don't have $\mathsf{f}(x)[x = 0] \to 1[x = 0]$.

$s[\pi] \to_{\mathsf{rule}} t[\psi]$ if

- π is satisfiable and $\psi = \pi$.
- $s = C[\ell\sigma]$ and $t = C[r\sigma]$ with $\rho : \ell \to r\ [\varphi] \in \mathcal{R}$
- $\mathrm{Dom}(\sigma) = \mathcal{V}(\ell, r, \varphi)$
- $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}(\pi) \cup \mathcal{V}al$
- $\models_{\mathcal{M}} (\pi \Rightarrow \varphi\sigma)$

Value-free-pattern LCTRSs

Definition

A rewrite rule $\ell \to r\ [\varphi]$ has *value-free-pattern* if ℓ does not contain value. An LCTRS $\mathcal{R}$ is value-free-pattern if so are all rules.

Lemma

For any rewrite rule ρ there exists a value-free-pattern rewrite rule ρ' such that $\forall s, t.\ (s \to_\rho t$ iff $s \to_{\rho'} t)$.

Proof. This is because for any $\rho : C[v_1, \ldots, v_n] \to r[\varphi]$ (with all values $v_1, \ldots, v_n$ in LHS indicated), one can take $\rho' : C[x_1, \ldots, x_n] \to r[\varphi \wedge x_1 = v_1 \wedge \cdots \wedge x_n = v_n]$, which is value-free-pattern. □

Thus, restricting rules to value-free-pattern is not a essential restriction.

Conjecture

Suppose

- $\mathcal{R}$ has value-free-pattern,
- π is satisfiable, $p \in \mathrm{Pos}(s)$, $\rho \in \mathcal{R}$,
- for any $u \in [\![\ s[\pi]\ ]\!]$ there exists u' such that $u \to_{\rho,p} u'$, and
- $\{u' \mid u \in [\![\ s[\pi]\ ]\!], u \to_{\rho,p} u'\} = [\![\ t[\pi]\ ]\!]$.

Then, $s[\pi] \to_{\mathsf{rule},p} t[\pi]$.

Proof. Let $\rho : \ell \to r\ [\varphi] \in \mathcal{R}$. By satisfiability, $[\![\ s[\pi]\ ]\!] \neq \emptyset$. Thus, there exists $u \in [\![\ s[\pi]\ ]\!]$ and u', such that $u \to_{\rho,p} u'$. Thus, $u = C[\ell\sigma]_p$, $u' = C[r\sigma]_p$, $\{\sigma(x) \mid x \in \mathcal{LV}ar(\rho)\} \subseteq \mathcal{V}al$, and $\models_{\mathcal{M}} \varphi\sigma$.
By $u \in [\![\ s[\pi]\ ]\!]$, $u = s\gamma$ for some γ such that γ respects π. W.l.o.g. one can take u in such a way that $\gamma(x) \notin \mathcal{V}al$ for any $x \notin \mathcal{V}(\pi)$.

Thus, by $u = s\gamma$ and $u = C[\ell\sigma]_p$, we know $C[\ell\sigma]_p = s\gamma$. Since $p \in Pos(s)$, we can take $s = \hat{C}[s']_p$.

Thus $C[\ell\sigma]_p = \hat{C}[s']_p\gamma = \hat{C}\gamma[s'\gamma]_p$. Thus, $C = \hat{C}\gamma$ and $\ell\sigma = s'\gamma$. Then, since ℓ does not contain values, one can let $s' = \ell\sigma'$ for some σ'. Then, $\ell\sigma = s'\gamma = \ell\sigma'\gamma$ and $\sigma'(x) \in \mathcal{V} \cup \mathcal{V}al$ for $x \in \mathcal{LV}ar(\rho)$ and $s = \hat{C}[s'] = \hat{C}[\ell\sigma']$.
Let $x \in \mathcal{LV}ar(\rho)$. By $\sigma(x) \in \mathcal{V}al$ and $\sigma(x) = \gamma(\sigma'(x))$, we have either $\sigma'(x) \in \mathcal{V}$ or $\sigma'(x) \in \mathcal{V}al$.
In the former case, we can take $\sigma'(x) = x'$ for some $x' \in \mathcal{V}(\pi)$, because of the way we take u and $\gamma(\sigma'(x)) \in \mathcal{V}al$.

Next, do we have $\models_{\mathcal{M}} (\pi \Rightarrow \varphi\sigma')$???
For this, we have to show that, for any valuation ξ on $\mathcal{M}$, $\models_{\mathcal{M},\xi} \pi$ implies $\models_{\mathcal{M},\xi} \varphi\sigma'$.
Suppose $\models_{\mathcal{M},\xi} \pi$. Then $\models_{\mathcal{M}} \pi\xi$. Thus, we could take $u(= s\gamma)$ such that $\gamma(x) = \xi(x)$ for all $x \in \mathcal{V}(\pi)$.
From $\models_{\mathcal{M}} \varphi\sigma$, maybe we get $\models_{\mathcal{M}} \varphi\sigma'\gamma$.(?) (Then, we have $\models_{\mathcal{M},\xi} \varphi\sigma'$.)
Currrently, I don't know the conjecture holds, or still there is a further counterexample.

Concluding Remarks

From perspective of interpreting LCTRSs in TRSs:

- interpetation of rewrite steps on terms seems to be understood clearly.
- for interpetation of rewrite steps on constrained terms:
  - it seems there is a natural interpretation $[\![\cdot]\!] : \mathrm{CnstrTerm} \to \mathrm{TermSet}$.
  - equivalence relation $\sim$ on $\mathrm{CnstrTerm}$ is mapped to the identity relation on $\mathrm{TermSet}$.
  - binary relation $\to_{\mathsf{calc}}$ on $\mathrm{CnstrTerm}$ relates to a relation on $\mathrm{TermSet}$ but not so clear. Also, characterization of relation on $\mathrm{TermSet}$ in terms of $\mathrm{CnstrTerm}$ is not clear.
  - binary relation $\to_{\mathsf{rule}}$ on $\mathrm{CnstrTerm}$ relates to a relation on $\mathrm{TermSet}$ but not so clear. Also, characterization of relation on $\mathrm{TermSet}$ in terms of $\mathrm{CnstrTerm}$ is unclear.
- Some related questions
  - What is the expressivity of $\mathrm{CnstrTerm}$? I.e., when a term set is expressed by a constrained term?
  - Is $\cdot \sim \cdot$ decidable? (YES $\Rightarrow$ [Kojima & Nishida, PRO2023]) More generally, what kinds of predicates on $\mathrm{TermSet}$ is computationally solved by means of $\mathrm{CnstrTerm}$?